ON THE PROPERTIES OF POLYNOMIALS SATISFYING A LINEAR DIFFERENTIAL EQUATION: PART I*

BY

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Introduction. Sequences of polynomials such as the Legendre, the Laguerre and the Hermite polynomials appeared in mathematics many years ago, and their properties have been investigated by numerous people. They satisfy simple difference equations, and also are solutions of linear differential equations of second order. The expansion problems (in the complex plane) associated with them are not so old. For the Legendre polynomials the region of convergence was determined by C. Neumann.† More recently the convergence regions for the Laguerre and Hermite polynomials were treated by O. Volk.‡ The paper of Volk considers, more generally, the boundary value problem (in the complex domain) for a second-order linear differential equation, not restricting attention to polynomials. The nth-order equation has since been treated, as a boundary value problem, by L. Bristow.§

Up to the present, however, there has been no general study of the properties of polynomials satisfying a linear differential equation of order higher than two. The present paper has in view such an investigation. There is another aspect to our treatment. In an earlier work we considered the properties of arbitrary *sets* of polynomials,|| associating with each set a linear differential equation, usually of infinite order. We obtained certain *formal* properties, whose complete justification required convergence proofs. The present paper deals with these matters for the case of a finite order equation.

§1 is preliminary: we state two theorems of Perron, and prove a corollary that is of use later. §2 introduces a fundamental differential equation whose polynomial solutions $\{y_n(x)\}$ we investigate, as well as the entire function

* Presented to the Society, December 27, 1929, under the title *The polynomial solutions of linear differential equations*; *Expansions*; received by the editors May 10, 1932.

† *Über die Entwicklung einer Funktion mit imaginärem Argument nach den Kugelfunktionen 1. und 2. Art*, Halle, 1862.

‡ *Über die Entwicklung von Funktionen einer komplexen Veränderlichen nach Funktionen, die einer linearen Differentialgleichung zweiter Ordnung mit einem Parameter genügen*, Mathematische Annalen, vol. 86 (1922), pp. 296–316.

§ *Expansion theory associated with linear differential equations and their regular singular points*, these Transactions, vol. 33 (1931), pp. 455–474.

|| *On sets of polynomials and associated linear functional operators and equations*, American Journal of Mathematics, vol. 53 (1931), pp. 15–38. We shall refer to this paper throughout as *Sets*.

solutions $\{\mathcal{D}_n(t)\}$ of the *dual* equation. §3 deals with general theorems on *sets*; §§ 4 and 5 give inequalities and resulting theorems of expansion; and in §6 we obtain certain biorthogonality relations and differential equations for functions allied to $\{y_n(x)\}$ and $\{\mathcal{D}_n(t)\}$.

The important problem of expansions in the polynomials $\{y_n(x)\}$ has hardly been touched. It demands considerations of another order from those of the present paper. Accordingly, we postpone its treatment to Part II.

1. **Preliminary: The Perron theorems.** We have need of the following two theorems (A and B) due to Perron:*

THEOREM A. *Consider the rth-order difference equation*

$$a_{i0}x_i + a_{i1}x_{i+1} + \cdots + x_{i+r} = 0 \qquad (i = 0, 1, \cdots). \tag{i}$$

Let $\lim_{i=\infty} a_{ij}$ *exist,* $=a_j, j=0, 1, \cdots, r-1$, *and let* $q_1, \cdots, q_k$ *be the distinct absolute values of the roots of the characteristic equation*

$$a_0 + a_1z + \cdots + z^r = 0.$$

Let e_m = *the number of zeros of absolute value* $q_m(e_1+ \cdots +e_m=r)$. *Then if* $a_{i0}\neq 0$ *for all i, there is a fundamental set of r solutions divided into k classes, such that the mth class contains* e_m *of these, and these* e_m *solutions satisfy the condition* $\limsup |x_n|^{1/n}=q_m$.

THEOREM B. *In the system of equations in infinitely many unknowns*

$$\sum_{n=0}^{\infty}(a_n + b_{in})x_{i+n} = c_i \qquad (i = 0, 1, \cdots), \tag{ii}$$

let the following conditions hold:

$$a_0 + b_{i0} \neq 0 \quad (i = 0, 1, \cdots); \quad \limsup |c_i|^{1/i} \leqq 1;$$
$$|b_{in}| \leqq k_i\theta^n, 0 < \theta < 1; \quad \lim_{i=\infty} k_i = 0;$$
$$F(z) = \sum_{0}^{\infty} a_nz^n \text{ is analytic in } |z| \leqq 1.$$

If in $|z|\leqq 1, F(z)$ *has n zeros (multiple roots counted multiply), then the general solution of* (ii) *satisfying the condition* $\limsup |x_s|^{1/s}\leqq 1$ *contains n arbitrary constants.*

It is not apparent from the statement of Theorem A that a solution $\{x_n\}$ (not $\equiv 0$) cannot be formed for which $\limsup |x_n|^{1/n}<\min (q_1, \cdots, q_k)$. As we need this fact, we shall establish

* *Über Summengleichungen und Poincarésche Differenzengleichungen*, Mathematische Annalen, vol. 84 (1921), pp. 1–15.

LEMMA 1.* *Under the hypotheses of Theorem* A *there is no solution not identically zero for which* $\limsup |x_n|^{1/n} < \min(q_1, \cdots, q_k)$.

Regarding $x=(x_0, x_1, \cdots)$ as a vector and $L[x]$ as the vector operator that carries x into the vector y with ith component

$$y_i = a_{i0}x_i + a_{i1}x_{i+1} + \cdots + x_{i+r},$$

let us determine an operator M that is inverse to L: $ML[x] \equiv x$. Let the ith component of $M[x]$ be

$$m_{i,i}x_i + m_{i,i+1}x_{i+1} + \cdots + m_{i,i+n}x_{i+n} + \cdots .$$

Then we are to have, identically in the $\{x_i\}$,

$$\sum_{i=s}^{\infty} m_{si}(a_{i0}x_i + a_{i1}x_{i+1} + \cdots + x_{i+r}) = x_s.$$

This gives the equations

$$\text{(a)}\quad \begin{aligned} &m_{ss}a_{s0} = 1, \\ &m_{ss}a_{si} + m_{s,s+1}a_{s+1,i-1} + \cdots + m_{s,s+i}a_{s+i,0} = 0 \qquad (i = 1, \cdots, r); \\ &m_{s,s+j}a_{s+j,r} + m_{s,s+j+1}a_{s+j+1,r-1} + \cdots + m_{s,s+j+r}a_{s+j+r,0} = 0 \\ &\qquad\qquad (j = 1, 2, \cdots). \end{aligned}$$

Since $a_{i0} \neq 0$ for all i, the quantities m_{ij} exist and are unique. M is then determined. It remains to consider convergence. Let s *be fixed*, and set

$$\text{(b)}\qquad n_k = m_{s,s+k}, \quad b_{k,r-i} = a_{s+k,i}.$$

Then we have the equations

$$\text{(c)}\qquad b_{i0}n_i + b_{i+1,1}n_{i+1} + \cdots + b_{i+r,r}n_{i+r} = 0 \qquad (i = 0, 1, \cdots)$$

for $n_0, n_1, \cdots$. It is easily verified that the conditions of Theorem A hold for (c), so that for every solution $\{n_k\}$ of (c) we have $\limsup |n_k|^{1/k} \leqq \max(|t_1|, \cdots, |t_r|)$, where $t_1, \cdots, t_r$ are the zeros of $1 + a_{r-1}t + \cdots + a_0 t^r = 0$. Now if $\min(q_1, \cdots, q_k) = 0$ the lemma is vacuously true. We may then assume that $\min(q_1, \cdots, q_k) = \lambda > 0$, in which case $a_0 \neq 0$. Then $t_1, \cdots, t_r$ are the reciprocals of the roots of the characteristic equation of (i), so that $\limsup |n_k|^{1/k} \leqq 1/\lambda$. To $\epsilon > 0$ we have $|m_{s,s+k}| \leqq K_s(\epsilon)/(\lambda - \epsilon)^k$ for all s.

Now suppose a solution $\{x_n\}$ of (i) exists such that $\limsup |x_n|^{1/n} = \delta < \lambda$. Then to $\epsilon' > 0$ we have $|x_n| < C(\epsilon')(\delta + \epsilon')^n$. Let $\gamma = \max(|a_{i0}|, \cdots, |a_{ir}|)$ for all i. Then

* A statement, without proof, of this lemma is given in Nörlund, *Differenzenrechnung*, 1924, p. 309.

$$| \{L[x]\}_i | = | a_{i0}x_i + \cdots + x_{i+r} |$$
$$\leqq C\gamma[(\delta + \epsilon')^i + \cdots + (\delta + \epsilon')^{i+r}] = C\gamma H(\delta + \epsilon')^i,$$

where the definition of H is obvious. Therefore

$$| \{M[L[x]]\}_s | \leqq K_s CH\gamma\left[\frac{(\delta + \epsilon')^s}{(\lambda - \epsilon)^0} + \frac{(\delta + \epsilon')^{s+1}}{(\lambda - \epsilon)} + \cdots\right]$$
$$= K_s CH\gamma(\delta + \epsilon')^s \sum_{t=0}^{\infty} \left(\frac{\delta + \epsilon'}{\lambda - \epsilon}\right)^t.$$

On choosing ϵ, ϵ' small enough the infinite geometric series converges. Hence, when we substitute $L[x]$ into M, forming $ML[x]$, and in the sth component combine coefficients of the same x_i's, we obtain an absolutely convergent series; the process is then legitimate. But $L[x] = (0, 0, \cdots)$, and since $ML[x] \equiv x$, it follows that $x = (0, 0, \cdots)$. This proves the lemma.

2. **Solutions of a differential equation and its dual.** Our principal aim is the study of the polynomial solutions of the kth-order linear differential equation

$$L[y(x)] \equiv L_0(x)y(x) + L_1(x)y'(x) + \cdots + L_k(x)y^{(k)}(x) = \lambda y(x), \tag{1}$$

where

$$L_i(x) = l_{i0} + l_{i1}x + \cdots + l_{ii}x^i \qquad (i = 0, 1, \cdots, k) \tag{2}$$

is a polynomial of degree not exceeding i and λ is a parameter, and of its dual equation (soon to be defined). We *define* λ_n by

$$\lambda_n = l_{00} + nl_{11} + n(n-1)l_{22} + \cdots + n(n-1)\cdots(n-k+1)l_{kk}. \tag{3}$$

THEOREM 1. *If* $\lambda_m \neq \lambda_n$, $m \neq n$, and if $l_{kk} \neq 0$, the equation*

$$L[y(x)] = \lambda y(x) \tag{4}$$

has an entire function solution ($\neq 0$) if and only if λ has one of the values $\lambda = \lambda_0, \lambda_1, \cdots$; and when $\lambda = \lambda_n$ there is just one entire function solution, namely a polynomial $y_n(x)$ of degree exactly n.

To demonstrate this, substitute into (2) the power series $y(x) = \sum_0^\infty y_n x^n$. On equating coefficients we find the following equations for the y_i:

$$(\lambda_n - \lambda)y_n + \sigma_{n,n+1}y_{n+1} + \sigma_{n,n+2}y_{n+2} + \cdots + \sigma_{n,n+k}y_{n+k} = 0 \qquad (n = 0, 1, \cdots), \tag{5}$$

* If $l_{kk} = 0$ or $\lambda_m = \lambda_n$ for some $m \neq n$, it is necessary to modify some of our later arguments, and we leave such considerations out of the present paper.

where

$$\begin{aligned}
\sigma_{n,n+1} &= l_{10}(n+1) + l_{21}(n+1)n + \cdots + l_{k,k-1}(n+1)n\cdots(n-k+2),\\
\sigma_{n,n+2} &= l_{20}(n+2)(n+1) + l_{31}(n+2)(n+1)n + \cdots\\
&\qquad + l_{k,k-2}(n+2)(n+1)\cdots(n-k+3),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\sigma_{n,n+k} &= l_{k0}(n+k)(n+k-1)\cdots(n+1).
\end{aligned} \tag{6}$$

For definiteness let us suppose that* $l_{k0}\neq 0$. We then obtain the difference equation

$$\frac{\lambda_n-\lambda}{\sigma_{n,n+k}}y_n + \frac{\sigma_{n,n+1}}{\sigma_{n,n+k}}y_{n+1} + \cdots + y_{n+k} = 0 \quad (n = 0, 1, \cdots), \tag{5'}$$

with

$$\lim_{n=\infty}\frac{\lambda_n-\lambda}{\sigma_{n,n+k}} = \frac{l_{kk}}{l_{k0}}, \quad \lim_{n=\infty}\frac{\sigma_{n,n+i}}{\sigma_{n,n+k}} = \frac{l_{k,k-i}}{l_{k0}} \quad (i = 1, \cdots, k-1). \tag{7}$$

The characteristic equation of (5′) is

$$l_{kk} + l_{k,k-1}t + \cdots + l_{k0}t^k = 0 \qquad (l_{kk}\neq 0). \tag{8}$$

Let $\alpha(>0)$ be the least absolute value of the roots of (5′). Then, by Lemma 1, for *every* solution $\{y_n\}$ we have $\lim\sup|y_n|^{1/n}\geqq\alpha$ provided $\lambda\neq\lambda_0, \lambda_1, \cdots$. Hence in this case $y(x)=\sum_0^\infty y_nx^n$, with radius of convergence $\leqq 1/\alpha$, cannot be an entire function (unless $y(x)\equiv 0$).

Now let $\lambda=\lambda_n$. A solution of equations (5) is seen to be $y_{n+1}=y_{n+2}=\cdots=0$, y_n arbitrary (but $\neq 0$), and $y_{n-1}, y_{n-2}, \cdots, y_0$ determined uniquely and successively (in view of $\lambda_m\neq\lambda_n$) from the $(n-1)$st, $(n-2)$d, $\cdots$, 0th equations of (5). Hence one entire function solution of (4) for($\lambda=\lambda_n$) is the polynomial

$$y_n(x) = y_{n0} + y_{n1}x + \cdots + y_{nn}x^n, \quad y_{nn}\neq 0. \tag{9}$$

To show that there is no other entire function solution, ignore the first $n+1$ equations of (5′). The system of equations remaining has the limits (7) for its coefficients. Moreover, *now* the coefficients of that y_s of lowest index in each equation is different from zero, so that Lemma 1 again applies, and the only entire function solution (for the modified system of equations) is the function *zero*. That is, the only entire function solution of (4) is one whose coefficients beyond x^n are all zero. And, since a polynomial solution of degree $\leqq n$ is unique, as we have seen, the theorem is established.

* If $l_{k0}=0$, (5) becomes a difference equation of order $<k$, but the same conclusion will follow.

DEFINITION. *The numbers* $\{\lambda_n\}$ *may be defined as the characteristic numbers of equation* (4).

Let us in (1) replace each $y^{(i)}$ by t^i, thus obtaining the function

$$L(t;\, x) = L_0(x) + L_1(x)t + \cdots + L_k(x)t^k; \tag{10}$$

or,

$$\begin{aligned} L(t;\, x) &= (l_{00} + l_{10}t + \cdots + l_{k0}t^k) + x(l_{11}t + l_{21}t^2 + \cdots \\ &\qquad + l_{k1}t^k) + \cdots + x^k(l_{kk}t^k) \\ &= \mathcal{L}_0(t) + \mathcal{L}_1(t)x + \cdots + \mathcal{L}_k(t)x^k. \end{aligned} \tag{10'}$$

Now in (10′) replace each power x^i by the derivative $\mathcal{D}^{(i)}(t)$ of a function* $\mathcal{D}(t)$. This gives us a differential expression which we term the *dual* of (1):

$$\mathcal{L}[\mathcal{D}(t)] \equiv \mathcal{L}_0(t)\mathcal{D}(t) + \mathcal{L}_1(t)\mathcal{D}'(t) + \cdots + \mathcal{L}_k(t)\mathcal{D}^{(k)}(t). \tag{11}$$

Evidently, (1) is the *dual* of (11). Similarly, we term equations (4) and

$$\mathcal{L}[\mathcal{D}(t)] \equiv \lambda\mathcal{D}(t) \tag{12}$$

dual equations.

THEOREM 2. *If* $\lambda_m \neq \lambda_n$, $m \neq n$, *and* $l_{kk} \neq 0$, *equation* (12) *has a formal power series solution about the origin if and only*† *if* $\lambda = \lambda_0, \lambda_1, \cdots$; *and when* $\lambda = \lambda_n$ *there is precisely one such power series*, $\mathcal{D}_n(t)(\neq 0)$, *and it is an entire function with an nth order zero at the origin.*

To prove this substitute the power series $\mathcal{D}(t) = \sum d_n t^n$ into (12) and equate coefficients. This yields the equations

$$(\lambda_n - \lambda)d_n + \alpha_{n,n-1}d_{n-1} + \alpha_{n,n-2}d_{n-2} + \cdots + \alpha_{n,n-k}d_{n-k} = 0 \quad (n = 0, 1, \cdots) \tag{13}$$

where

$$\begin{aligned} \alpha_{n,n-1} &= l_{10} + l_{21}(n-1) + l_{32}(n-1)(n-2) + \cdots \\ &\qquad + l_{k,k-1}(n-1)(n-2)\cdots(n-k+1), \\ \alpha_{n,n-2} &= l_{20} + l_{31}(n-2) + \cdots + l_{k,k-2}(n-2)\cdots(n-k+1), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{n,n-k} &= l_{k0}. \end{aligned} \tag{14}$$

If $\lambda \neq \lambda_0, \lambda_1, \cdots$ then successively we get $d_0 = d_1 = \cdots = 0$, so that the only formal power series solution is $\mathcal{D}(t) \equiv 0$. Now suppose $\lambda = \lambda_n$. Then $d_0 = d_1 = \cdots = d_{n-1} = 0$, d_n is arbitrary, and $d_{n+1}, d_{n+2}, \cdots$ are successively

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* See *Sets*, loc. cit., pp. 31–32.

† Consequently, we may say that equations (4) and (12) have the same characteristic numbers.

and uniquely determined (in terms of d_n). Hence there is just one formal solution, $\sum_{i=n}^{\infty} d_i t^i$. We proceed to show this solution is an entire function. The first $n+1$ equations of (13) drop out, giving us

$$(\lambda_s - \lambda_n)d_s + \alpha_{s,s-1}d_{s-1} + \cdots + \alpha_{s,s-k}d_{s-k} = 0 \qquad (s = n+1, n+2, \cdots);$$

and, on setting $r = s-k$, we get the difference equation

$$d_{r+k} + \frac{\alpha_{r+k,r+k-1}}{\lambda_{r+k} - \lambda_n} d_{r+k-1} + \cdots + \frac{\alpha_{r+k,r}}{\lambda_{r+k} - \lambda_n} d_r = 0 \tag{13'}$$
$$(r = n+1-k, n+2-k, \cdots),$$

with

$$\lim_{r=\infty} \frac{\alpha_{r+k,r+k-i}}{\lambda_{r+k} - \lambda_n} = 0 \qquad (i = 1, \cdots, k), \tag{15}$$

and with the characteristic equation

$$t^k = 0. \tag{16}$$

By* the Perron Theorem A, for every solution of (13′) we have lim sup $|d_r|^{1/r} = 0$; and this implies that $\mathcal{D}(t)$ is an entire function.

We can say even more:

COROLLARY. *The solution* $\mathcal{D}_n(t) = \sum_{i=n}^{\infty} d_{ni}t^i$ *corresponding to* $\lambda = \lambda_n$ *satisfies the inequality*

$$\limsup_{s=\infty} | \mathcal{D}_n^{(s)}(0) |^{1/s} \leqq \rho = \textit{maximum absolute value of the zeros of } L_k(x),$$

so that the function $\Delta_n(t)$ *defined by*

$$\Delta_n(t) = \sum_{i=n}^{\infty} i!\, d_{ni} t^i \tag{17}$$

has a radius of convergence at least equal to $1/\rho$.

To show this, let $d_{ni} = v_i/i!$; then (13′) leads to the difference equation

$$\begin{aligned} v_{r+k} &+ \frac{\alpha_{r+k,r+k-1}}{\lambda_{r+k} - \lambda_n}(r+k)v_{r+k-1} + \cdots \\ &+ \frac{\alpha_{r+k,r}}{\lambda_{r+k} - \lambda_n}(r+k)(r+k-1)\cdots(r+1)v_r = 0, \end{aligned} \tag{13''}$$

with

* If $l_{k0} = 0$, (13′) is a difference equation of order less than k, but the same conclusion follows.

$$\lim_{r=\infty} \frac{\alpha_{r+k,r+k-i}}{\lambda_{r+k}-\lambda_n}(r+k)(r+k-1)\cdots(r+k-i+1) = \frac{l_{k,k-i}}{l_{kk}}$$
$$(i = 1, \cdots, k),$$

and to the characteristic equation $L_k(t)=0$, whose largest zero is in absolute value ρ. Now the coefficient of v_r in (13″) is never zero.* Hence, from Theorem A, lim sup $|v_r|^{1/r} \leqq \rho$; and from this the corollary follows.

$\mathcal{D}_n(t)$ is the so-called Borel entire function associated with $\Delta_n(t)$ and the two are related by the following integral:

$$\mathcal{D}_n(t) = \frac{1}{2\pi i}\int_C \frac{e^{tu}}{u}\Delta_n\left(\frac{1}{u}\right)du, \tag{18}$$

where C is a closed contour surrounding the origin and lying wholly outside of $|u|=\rho$.

Similarly, if we set

$$Y_n(x) = 0!\,y_{n0} + 1!\,y_{n1}x + \cdots + n!\,y_{nn}x^n, \tag{19}$$

then $y_n(x)$ is the Borel entire function for $Y_n(x)$, and we have

$$y_n(x) = \frac{1}{2\pi i}\int_\Gamma \frac{e^{xu}}{u}\,Y_n\left(\frac{1}{u}\right)du, \tag{20}$$

Γ being a closed contour surrounding the origin.

DEFINITION. *An analytic function $f(t;\,x)$ is self-dual with respect to the above operators L, $\mathcal{L}$, operating respectively on the variables x and t, if*

$$L[f(t;\,x)] = \mathcal{L}[f(t;\,x)].$$

COROLLARY. *e^{tx} is a self-dual, and*

$$L[e^{tx}] = \mathcal{L}[e^{tx}] = e^{tx}L(t;\,x). \tag{21}$$

3. Associated sets of functions; the sets P_λ, Q_λ. Let us now consider the following parametric differential expressions corresponding to (1) and (11):

$$L_\lambda[y(x)] \equiv L[y(x)] - \lambda y(x), \tag{22}$$

$$\mathcal{L}_\lambda[\mathcal{D}(t)] \equiv \mathcal{L}[\mathcal{D}(t)] - \lambda\mathcal{D}(t). \tag{23}$$

Define the *set* of polynomials P_λ: $\{P_n(x;\,\lambda)\}$ by

$$\begin{aligned} P_n(x;\lambda) = L_\lambda[x^n] = (L_0(x)-\lambda)x^n + nL_1(x)x^{n-1} + \cdots \\ + n(n-1)\cdots(n-k+1)L_k(x)x^{n-k} \quad (n = 0, 1, \cdots), \end{aligned} \tag{24}$$

* That is, if $l_{k0}\neq 0$. Should $l_{k0}=0$ (here and hereafter), the remark of a previous footnote applies.

and the *set* of functions $\mathcal{P}_\lambda$: $\{\mathcal{P}_n(t;\lambda)\}$ by

$$\text{(25)}\quad \begin{aligned}\mathcal{P}_n(t;\lambda) &= \mathcal{L}_\lambda[t^n] = (\mathcal{L}_0(t)-\lambda)t^n + n\mathcal{L}_1(t)t^{n-1}+\cdots\\ &\quad + n(n-1)\cdots(n-k+1)\mathcal{L}_k(t)t^{n-k}\quad (n=0,1,\cdots).\end{aligned}$$

We see that $P_n(x;\lambda)$ is a polynomial in x, of degree n if $\lambda\neq\lambda_0, \lambda_1, \cdots$, and that $\mathcal{P}_n(t;\lambda)$ is a polynomial in t, of degree not exceeding $n+k$, with a zero at the origin of order at least n.

Definition. *Let $H(t;x)$ be a symbol for a formal power series in t with coefficients that are formal power series in x, so that when it is expressed formally as a power series in x, the coefficients are (formal) power series in t:*

$$H(t;x)\sim\sum_0^\infty h_n(x)t^n/n!\sim\sum_0^\infty \mathcal{H}_n(t)x^n/n!\qquad (h_n(x),\ \mathcal{H}_n(t)\ \textit{power series}).$$

Then we say that the two sets of functions $\{h_n(x)\}$, $\{\mathcal{H}_n(t)\}$ $(n=0, 1, \cdots)$ are associated sets.

Lemma 2. *The two sets P_λ, $\mathcal{P}_\lambda$ given in* (24, 25) *are associated sets, and, for* all x and t,*

$$\text{(26)}\quad \sum_{n=0}^\infty P_n(x;\lambda)t^n/n! = \sum_{n=0}^\infty \mathcal{P}_n(t;\lambda)x^n/n! = e^{tx}L_\lambda(t;x) = (\textit{definition})P(t,x;\lambda),$$

where

$$\text{(27)}\qquad L_\lambda(t;x)\equiv L(t;x)-\lambda.$$

(26) follows from (24), (25) and (21). The convergence of (26) is immediate.

If we expand (24), we obtain

$$\text{(28)}\quad P_n(x;\lambda) = (\lambda_n-\lambda)x^n+\sigma_{n-1,n}x^{n-1}+\sigma_{n-2,n}x^{n-2}+\cdots+\sigma_{n-k,n}x^{n-k},$$

where the σ_{ij} are given by (6).

In our theory of sets of polynomials† we considered the *multiplication* of sets. Thus, if P: $\{P_n(x)\}$, Q: $\{Q_n(x)\}$ are any two sets, where

$$P_n(x) = p_{n0}+p_{n1}x+\cdots+p_{nn}x^n,\ Q_n(x) = q_{n0}+q_{n1}x+\cdots+q_{nn}x^n,$$

then PQ is the set $\{PQ_n(x)\}$, where

$$PQ_n(x) = p_{n0}Q_0(x)+p_{n1}Q_1(x)+\cdots+p_{nn}Q_n(x)\quad (n=0,1,\cdots).$$

In particular, a set Q is the *inverse* of P if $PQ=I$ where I is the identity set: $I_n(x)=x^n$. It is easy to see that an inverse Q exists if and only if $P_n(x)$ is of

* (26) is formally true in the general theory of sets of polynomials.

† *Sets*, p. 16.

degree exactly n for every n; and in this case Q is unique, and P is also the inverse of Q: $QP=I$.

Now $P_n(x;\lambda)$ is of degree exactly $n(\lambda\neq\lambda_0, \lambda_1, \cdots)$. Hence the set P_λ possesses an *inverse* set Q_λ: $\{Q_n(x;\lambda)\}$. Since $P_\lambda Q_\lambda=I$, there follows from (28)

LEMMA 3. *The set* $\{Q_n(x;\lambda)\}$ *satisfies the difference equation*

$$(\lambda_n-\lambda)Q_n(x;\lambda)+\sigma_{n-1,n}Q_{n-1}(x;\lambda)+\cdots+\sigma_{n-k,n}Q_{n-k}(x;\lambda)=x^n \qquad (n=0,1,\cdots). \tag{29}$$

Examination of the first few $Q_n(x;\lambda)$'s suggests

LEMMA 4. $Q_n(x;\lambda)$ *is a rational function of* λ, *having at most* simple poles at* $\lambda=\lambda_0, \lambda_1, \cdots, \lambda_n$.

This is true for $n=0$, 1 as is readily seen. The Lemma then follows by induction from (29).

From (24) we find that

$$L_\lambda[Q_n(x;\lambda)]=QP_n(x;\lambda)=x^n \qquad (n=0,1,\cdots). \tag{30}$$

If we multiply through by $t^n/n!$ and sum formally from $n=0$ to ∞, we obtain

$$L_\lambda[Q(t,x;\lambda)]=e^{tx} \tag{31}$$

where

$$Q(t,x;\lambda)=\sum_{n=0}^{\infty}Q_n(x;\lambda)t^n/n!. \tag{32}$$

We proceed to show that† (32) is uniformly convergent in t and x in some region, thus making (31) valid.

On dividing (29) through by $\sigma_{n-k,n}$, we obtain a system of equations for $Q_0, Q_1, \cdots$ satisfying all the hypotheses of Perron Theorem B, the function $F(z)$ here being $L_k(z)/l_{k0}$, provided $|x|\leqq 1$. If $L_k(z)$ does not have all its k zeros in $|z|\leqq 1$, we cannot apply Theorem B to all the solutions of the system in question. This difficulty can be overcome by modifying the system as follows:

Let ρ again denote the largest absolute value of the zeros of $L_k(z)$ and set $Q_s(x;\lambda)=\rho^s R_s(x;\lambda)$. We then obtain for $\{R_s\}$ the system of equations

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* Individual $Q_n(x;\lambda)$'s may fail to have a pole at some of these points; e.g., $Q_1(x;\lambda)$ will not have λ_0 as a pole if $l_{10}=0$. It is however easy to prove that $Q_n(x;\lambda)$ always has λ_n as a pole (i.e., if x is not given special values).

† $Q(t,x;\lambda)$ is a function of λ as well; we must therefore avoid such values of λ as will make $Q(t,x;\lambda)$ singular.

$$(\alpha)\quad R_s + \frac{\sigma_{k+s,s+1}}{\sigma_{s,s+k}}\rho R_{s+1} + \cdots + \frac{\lambda_{k+s} - \lambda}{\sigma_{s,s+k}}\rho^k R_{s+k} = \frac{x^{s+k}}{\sigma_{s,s+k}\rho^s} \quad (s = 0, 1, \cdots).$$

This system satisfies the conditions of Theorem B, if we restrict x to lie in $|x| \leqq \rho$, and the function $F(z)$ is here $(1/l_{k0})L_k(z\rho)$. Now the characteristic roots all lie in $|z| \leqq 1$ so that by Theorem B, the general solution of (α) for which $\limsup |R_s|^{1/s} \leqq 1$ contains k arbitrary constants, exactly as many as enter into the general solution of (α). Hence, *every* solution $\{R_s\}$ of (α) satisfies the condition $\limsup |R_s|^{1/s} \leqq 1$. We thus have

COROLLARY 1. *For* $|x| \leqq \rho$,

$$(33)\qquad \limsup_{n=\infty} |Q_n(x;\lambda)|^{1/n} \leqq \rho \qquad (\lambda \neq \lambda_0, \lambda_1, \cdots, \lambda_n).$$

We have, moreover, *uniformly** in $|x| \leqq \rho$,

$$(33')\qquad |Q_n(x;\lambda)| \leqq C(\rho + \epsilon)^n.$$

Here $\epsilon > 0$ is arbitrary, and C does not depend on x.

If we had defined $R_s(x;\lambda)$ by $Q_s(x;\lambda) = \delta^s R_s(x;\lambda)$, $\delta \geqq \rho$, the argument made above would continue to hold, giving us

COROLLARY 2. *For* $|x| \leqq \delta$, $\delta(\geqq \rho)$ *arbitrary*,

$$(34)\qquad \limsup_{n=\infty} |Q_n(x;\lambda)|^{1/n} \leqq \delta \qquad (\lambda \neq \lambda_0, \lambda_1, \cdots, \lambda_n);$$

and

$$(34')\qquad |Q_n(x;\lambda)| \leqq C_\delta(\delta + \epsilon)^n$$

uniformly in $|x| \leqq \delta$. *Here* $\epsilon > 0$ *is arbitrary, and* C_δ *depends only on* ϵ *and* δ.

Since δ may be chosen arbitrarily large we have

THEOREM 3. *The function* $Q(t, x; \lambda)$ *given by* (32) *is analytic*† *in* t, x, λ; *it is an entire function in* t *and* x, *and its singularities in* λ *are at most at the points* $\lambda = \lambda_0, \lambda_1, \cdots$. *Moreover* (31) *holds for every* t, x, λ $(\lambda \neq \lambda_0, \lambda_1, \cdots)$.

Let us return to the theory of *sets*.‡ We have there given the

DEFINITION. *A triangular set of functions* $\{\mathcal{P}_n(t)\}$, $n = 0, 1, \cdots$, *is a set of formal power series in* t *such that* $\mathcal{P}_n(t)$ *begins with a power of* t *not less than* n:

$$\mathcal{P}_n(t) \sim \pi_{nn}t^n + \pi_{n,n+1}t^{n+1} + \cdots.$$

* The uniformity can be established from the Perron proofs.

† Relation (34′) is also uniform in every bounded λ-region, the points $\lambda = \lambda_0, \lambda_1, \cdots$ being deleted.

‡ *Sets*, p. 32.

For such sets multiplication is defined as follows:

$$\mathfrak{P}\mathfrak{Q} \equiv \{\mathfrak{P}\mathfrak{Q}_n(t)\}: \ \mathfrak{P}\mathfrak{Q}_n(t) \sim \pi_{nn}\mathfrak{Q}_n(t) + \pi_{n,n+1}\mathfrak{Q}_{n+1}(t) + \cdots \quad (n = 0, 1, \cdots).$$

If, in particular, $\mathfrak{P}\mathfrak{Q} = \mathfrak{I}$ where $\mathfrak{I}$: $\{\mathfrak{I}_n(t) = t^n\}$ is the *identity* set, then $\mathfrak{Q}$ is the *inverse* of $\mathfrak{P}$. Such $\mathfrak{Q}$ exists if and only if $\pi_{nn} \neq 0$, $n = 0, 1, \cdots$, and then $\mathfrak{Q}$ is unique, and $\mathfrak{P}$ is the inverse of $\mathfrak{Q}$.

Let P: $\{P_n(x)\}$ be a polynomial set, and $\mathfrak{P}$: $\{\mathfrak{P}_n(t)\}$ the associated set. ($\mathfrak{P}$ is then a triangular set.) On setting

$$P_n(x) = p_{n0} + p_{n1}x + \cdots + p_{nn}x^n, \quad \mathfrak{P}_n(t) \sim \pi_{nn}t^n + \pi_{n,n+1}t^{n+1} + \cdots,$$

it is seen that the property of being associated sets is equivalent to the relations

$$(35) \quad \begin{aligned} \pi_{nn} &= p_{nn}, \\ \pi_{n,n+i} &= p_{n+i,n}/((n+i)(n+i-1)\cdots(n+1)) \quad (i = 1, 2, \cdots). \end{aligned}$$

We can establish the following general theorem on sets:

THEOREM 4. (a) *Let P, $\mathfrak{P}$ be associated sets, and Q, $\mathfrak{Q}$ their respective inverses. Then Q, $\mathfrak{Q}$ are associated sets.*

(b) *If P, Q are inverse sets, and $\mathfrak{P}$, $\mathfrak{Q}$ their associated sets, then $\mathfrak{P}$, $\mathfrak{Q}$ are inverse sets.*

Consider (a). Let

$$Q_n(x) = q_{n0} + \cdots + q_{nn}x^n, \qquad \mathfrak{Q}_n(t) \sim \kappa_{nn}t^n + \kappa_{n,n+1}t^{n+1} + \cdots.$$

From $PQ = I$ and $\mathfrak{P}\mathfrak{Q} = \mathfrak{I}$ we obtain the relations

$$p_{n0}Q_0(x) + \cdots + p_{nn}Q_n(x) = x^n, \quad \pi_{nn}\mathfrak{Q}_n(t) + \pi_{n,n+1}\mathfrak{Q}_{n+1}(t) + \cdots \sim t^n,$$

$$(\alpha) \quad p_{nn}q_{n,n-i} + p_{n,n-1}q_{n-1,n-i} + \cdots + p_{n,n-i}q_{n-i,n-i} = \begin{cases} 1, & i = 0, \\ 0, & i = 1, \cdots, n; \end{cases}$$

$$(\beta) \quad \pi_{nn}\kappa_{n,n+i} + \pi_{n,n+1}\kappa_{n+1,n+i} + \cdots + \pi_{n,n+i}\kappa_{n+i,n+i} = \begin{cases} 1, & i = 0, \\ 0, & i = 1, 2, \cdots. \end{cases}$$

The theorem will be proved if we establish that

$$(\text{i}) \quad \begin{aligned} \kappa_{nn} &= q_{nn}, \\ \kappa_{n,n+i} &= q_{n+i,n}/((n+i)\cdots(n+1)) \qquad (i = 1, 2, \cdots), \end{aligned}$$

or, that $s_{n,n+i} = \kappa_{n,n+i}$, $i = 0, 1, \cdots$, where

$$(\text{ii}) \quad \begin{aligned} s_{nn} &= q_{nn}, \\ s_{n,n+i} &= q_{n+i,n}/((n+i)\cdots(n+1)). \end{aligned}$$

In equations (α), substitute for $q_{n,n+i}$ its value in terms of $s_{n+i,n}$. Moreover, in the resulting equations, leave the first equation unchanged, replace n by

$n+1$ in the second, n by $n+2$ in the third, and so on. This gives the following equations (α'), equivalent to equations (α):

$$(\alpha') \quad \pi_{n+i,n+i}s_{n,n+i} + \pi_{n+i-1,n+i}s_{n,n+i-1} + \cdots + \pi_{n,n+i}s_{nn} = \begin{cases} 1, & i = 0, \\ 0, & i = 1, 2, \cdots. \end{cases}$$

We verify at once that $s_{nn} = \kappa_{nn}$, $s_{n,n+1} = \kappa_{n,n+1}$. Suppose $s_{n,n+j} = \kappa_{n,n+j}$ for $j = 0, 1, \cdots, i-1$. We shall then prove it for $j = i$, and the theorem will be demonstrated. Denote respectively by $E_{n0}, E_{n1}, \cdots; F_{n0}, F_{n1}, \cdots$ the left hand members of (α') and (β) for $i = 0, 1, \cdots$. Now form linear expressions in the s's and κ's, respectively:

$$E_i \equiv \pi_{n,n+i}E_{n+i,0} + \pi_{n,n+i-1}E_{n+i-1,1} + \cdots + \pi_{nn}E_{ni} = \pi_{n,n+i},$$

$$F_i \equiv \pi_{n,n+i}F_{n0} + \pi_{n+1,n+i}F_{n1} + \cdots + \pi_{n+i,n+i}F_{ni} = \pi_{n,n+i}.$$

The coefficient in E_i of $s_{n+i-p,n+i-p+q}$ is readily seen to be $\pi_{n,n+i-p}\,\pi_{n+i-p+q,n+i}$, and this is also the coefficient of $\kappa_{n+i-p,n+i-p+q}$ in F_i. Hence from $E_i = F_i$, and our induction assumption, it follows that $\pi_{nn}\,\pi_{n+i,n+i}\,s_{n,n+i} = \pi_{n+i,n+i}\pi_{nn}\kappa_{n,n+i}$. But $\pi_{nn} = p_{nn} \neq 0$, $n = 0, 1, \cdots$ (since P possesses an inverse). Therefore, $s_{n,n+i} = \kappa_{n,n+i}$ and the induction is complete.

To establish (b), let $\mathfrak{Q}^*$ be the inverse of $\mathfrak{P}$. Then by (a), Q and $\mathfrak{Q}^*$ are associated sets. But the associate of a set is unique, so that $\mathfrak{Q} \equiv \mathfrak{Q}^*$.

Consider the associate sets P_λ, $\mathfrak{P}_\lambda$ of (26). The coefficient of t^n in $\mathfrak{P}_n(t;\lambda)$ is not zero for $\lambda \neq \lambda_0, \lambda_1, \cdots$. Hence $\mathfrak{P}_\lambda$ possesses an inverse $\mathfrak{Q}_\lambda$: $\{\mathfrak{Q}_n(t;\lambda)\}$, and by Theorem 4 we have the

COROLLARY. *Q_λ and $\mathfrak{Q}_\lambda$ are associated sets, and*

$$(36) \qquad Q(t, x; \lambda) = \sum_{n=0}^{\infty} Q_n(x; \lambda)t^n/n! = \sum_{n=0}^{\infty} \mathfrak{Q}_n(t; \lambda)x^n/n!,$$

the series converging uniformly in every bounded x, t, λ region (on deleting the points $\lambda_0, \lambda_1, \cdots$).

From the definition (25) of $\mathfrak{P}_\lambda$ we have (using (14))

$$(37) \quad \mathfrak{P}_n(t;\lambda) = (\lambda_n - \lambda)t^n + \alpha_{n+1,n}t^{n+1} + \alpha_{n+2,n}t^{n+2} + \cdots + \alpha_{n+k,n}t^{n+k}.$$

On equating corresponding coefficients in $\mathfrak{P}_\lambda\mathfrak{Q}_\lambda = \mathfrak{I}$ there results the following difference equation for $\mathfrak{Q}_n(t;\lambda)$:

$$(38) \quad (\lambda_n - \lambda)\mathfrak{Q}_n(t;\lambda) + \alpha_{n+1,n}\mathfrak{Q}_{n+1}(t;\lambda) + \cdots + \alpha_{n+k,n}\mathfrak{Q}_{n+k}(t;\lambda) = t^n \qquad (n = 0, 1, \cdots).$$

Now $\mathfrak{Q}_\lambda$ is the inverse of $\mathfrak{P}_\lambda$, so that we have

$$(39) \qquad \mathcal{L}_\lambda[\mathfrak{Q}_n(t;\lambda)] = t^n,$$

(40) $$\mathcal{L}_\lambda[Q(t, x; \lambda)] = L_\lambda[Q(t, x; \lambda)] = e^{tx}. \text{ (See (31, 36).)}$$

On applying the operator $\mathcal{L}_\lambda$ to (32), we then have $e^{tx} = \sum_0^\infty Q_n(x;\lambda)\mathcal{L}_\lambda[t^n]/n!$, or

(41) $$e^{tx} = \sum_{n=0}^{\infty} Q_n(x; \lambda)\, \mathcal{P}_n(t; \lambda)/n!.$$

Similarly we obtain

(42) $$e^{tx} = \sum_{n=0}^{\infty} \mathfrak{Q}_n(t; \lambda) P_n(x; \lambda)/n!.$$

The series in (41, 42) converge uniformly for all t, x, λ bounded (on deleting $\lambda = \lambda_0, \lambda_1, \cdots$).

From the way in which (41) and (42) were established it is clear that they hold *formally* in the general theory of sets. Indeed we can state the more general

THEOREM 5. *Let P be any polynomial set, and let $\mathfrak{Q}$ be the associate of the inverse of P. Then*

(43) $$e^{tx} \sim \sum_{n=0}^{\infty} P_n(x)\mathfrak{Q}_n(t)/n!.$$

For, let L be the differential operator* (in general of infinite order) that carries the identity set I into P: $L[x^n] = P_n(x)$, and let Q be the inverse of P, so that $L[Q_n(x)] = x^n$. Since Q and $\mathfrak{Q}$ are associate,

$$Q(t, x) \sim \sum_0^\infty Q_n(x)t^n/n! \sim \sum_0^\infty \mathfrak{Q}_n(t)x^n/n!.$$

Then,

$$L[Q(t, x)] \sim e^{tx} \sim \sum_n \mathfrak{Q}_n(t)L[x^n]/n! \sim \sum_n \mathfrak{Q}_n(t)P_n(x)/n!,$$

and this is (43).

If P is a polynomial set (if $\mathfrak{Q}$ is a triangular set) then $\mathfrak{Q}$ (P) is uniquely determined by (43). Hence we have the converse

THEOREM 6. *If P, $\mathfrak{Q}$ are any polynomial and triangular sets, respectively, satisfying* (43), *then $\mathfrak{Q}(P)$ is the associate of the inverse of $P(\mathfrak{Q})$.*

We can now complete Theorem 3. Define

$$R_n(t, x; \lambda) = (\lambda - \lambda_n)Q(t, x; \lambda).$$

* That L exists is established in *Sets*, p. 29.

Since $Q_i(x;\lambda)$ contains at most a simple pole at $\lambda=\lambda_n$, $R_n(t, x;\lambda)$ is analytic at $\lambda=\lambda_n$. $Q(t, x;\lambda)$ has then, at $\lambda=\lambda_n$, at most a pole of first order. Now

$$\mathcal{L}[Q(t, x;\lambda)] = \mathcal{L}_\lambda[Q] + \lambda Q = e^{tx} + \lambda Q(t, x;\lambda);$$

therefore $\mathcal{L}[R_n(t, x;\lambda)]=(\lambda-\lambda_n)e^{tx}+\lambda R_n(t, x;\lambda)$, and (on letting $\lambda\to\lambda_n$) $\mathcal{L}[R_n(t, x;\lambda_n)]=\lambda_n R_n(t, x;\lambda_n)$. Now $R_n(t, x;\lambda_n)$ can be expanded about $t=0$. Hence by Theorem 2 it can differ from $\mathcal{D}_n(t)$ at most by a factor independent of t:$R_n(t, x;\lambda_n)=H_n(x)\mathcal{D}_n(t)$. Similarly, $L[R_n(t, x;\lambda_n)]=\lambda_n R_n(t, x;\lambda_n)$, and since $R_n(t, x;\lambda_n)$ is an entire function in x, we must have (Theorem 1)

(i) $$R_n(t, x;\lambda_n) = c_n y_n(x)\mathcal{D}_n(t) \quad (c_n = \text{constant}).$$

We agree from now on that in $y_n(x)$ and $\mathcal{D}_n(t)$ we shall choose the coefficients of x^n, t^n respectively to be unity:

(ii) $$y_{nn} = 1,\ d_{nn} = 1.$$

From (29) we see that the coefficient of x^n in $(\lambda-\lambda_n)Q_n(x;\lambda)$ is -1, and since

$$R_n(t, x;\lambda_n) = \sum_{i=n}^{\infty}\left\{\lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)Q_n(x;\lambda)\right\}t^n/n!,$$

the coefficient of $x^n t^n$ in $R_n(t, x;\lambda_n)$ is $-1/n!$. Hence, by (i, ii), $c_n=-1/n!$.

Now let γ_n be a closed contour in the λ-plane surrounding the point $\lambda=\lambda_n$ but containing in its interior and on its boundary no other λ_i. Then

$$\frac{1}{2\pi i}\int_{\gamma_n} Q(t, x;\lambda)d\lambda = \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)Q(t, x;\lambda) = R_n(t, x;\lambda_n).$$

THEOREM 7. *At each of the points* $\lambda=\lambda_n$, $Q(t, x;\lambda)$ *has a simple pole with residue* $-y_n(x)\mathcal{D}_n(t)/n!$, *so that*

(44) $$\frac{1}{2\pi i}\int_{\gamma_n} Q(t, x;\lambda)d\lambda = -y_n(x)\mathcal{D}_n(t)/n!.$$

From (44) we have

$$-y_n(x)\mathcal{D}_n(t)/n! = \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)Q(t, x;\lambda).$$

On using (32), then,

(a) $$\begin{aligned} -y_n(x)\mathcal{D}_n(t)/n! &= \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)Q_n(x;\lambda)t^n/n! \\ &\quad + \lim_{\lambda\to\lambda_n}(\lambda-\lambda_n)Q_{n+1}(x;\lambda)t^{n+1}/(n+1)! + \cdots, \end{aligned}$$

so that on equating coefficients of t^n,

(b) $$-y_n(x)/n! = \lim_{\lambda\to\lambda_n} (\lambda - \lambda_n)Q_n(x;\lambda)/n!.$$

From (b) follows the relation

(c) $$y_n(x) = \frac{-1}{2\pi i}\int_{\gamma_n} Q_n(x;\lambda)d\lambda.$$

Using (29) in (b), we can express $y_n(x)$ in terms of the Q_i's:

(44′) $$y_n(x) = x^n - \left\{\sigma_{n-1,n}Q_{n-1}(x;\lambda_n) + \cdots + \sigma_{n-k,n}Q_{n-k}(x;\lambda_n)\right\}.$$

This relation can be reversed rather simply as follows:

Equating like powers of t in (a) gives

$$-y_n(x)d_{n,n+i}/n! = \lim_{\lambda\to\lambda_n} (\lambda - \lambda_n)Q_{n+i}(x;\lambda)/(n+i)!,$$

so that $y_n(x)$ is, with a certain constant factor, the residue of $Q_{n+i}(x;\lambda)$ at $\lambda=\lambda_n$. Hence $y_{n-i}(x)$ is, to within a factor, the residue of $Q_n(x;\lambda)$ at $\lambda=\lambda_{n-i}$, and there exist certain constants $q_{n0}, q_{n1}, \cdots, q_{nn}$ (independent of x and λ) such that

(44″) $$Q_n(x;\lambda) = q_{n0}\frac{y_0(x)}{\lambda-\lambda_0} + q_{n1}\frac{y_1(x)}{\lambda-\lambda_1} + \cdots + q_{nn}\frac{y_n(x)}{\lambda-\lambda_n}\cdot$$

The q_{ni}'s can be found successively from recurrence relations.

The relation (44) suggests the partial fraction expansion

(45) $$Q(t, x;\lambda) = \sum_{n=0}^{\infty}\frac{y_n(x)\mathcal{D}_n(t)}{(\lambda_n-\lambda)n!}\cdot$$

Assuming (45) to converge uniformly, and applying the operator L_λ, we obtain

(46) $$e^{tx} = \sum_{n=0}^{\infty} y_n(x)\mathcal{D}_n(t)/n!.$$

We shall establish the validity of (45, 46) after we have developed some necessary inequalities. Let us observe, however, that (46) is of type (43) so that by Theorem 6 (and this holds for the general theory of sets) we have the

COROLLARY. *The sets* $\{y_n(x)\}$, $\{\mathcal{D}_n(t)\}$ *are respectively the associates of each other's inverse.*

4. Some inequalities for $y_n(x)$, $\mathcal{D}_n(t)$. Let λ_n be defined as in (3).

LEMMA 5. *For all n and for all $i>0$,*

(47) $$|\lambda_{n+i}-\lambda_n| \geqq C\left[\frac{k}{1!}n^{k-1}i+\frac{k(k-1)}{2!}n^{k-2}i^2+\cdots+\frac{k!}{k!}i^k\right],$$

where C is a positive constant independent of both n and i.

To show this, we can write

(a) $$\lambda_{n+i}-\lambda_n = B(n,i)+l_{kk}[A_{k-1,1}n^{k-1}i+A_{k-2,2}n^{k-2}i^2+\cdots+A_{0k}i^k]$$

where $B(n,i)$ is a polynomial in n and i of degree $<k$; and a simple calculation gives for the A_{rs} the values

$$A_{k-1,1}=k/1!,\ A_{k-2,2}=k(k-1)/2!,\cdots,A_{0k}=k!/k!.$$

The bracket on the right hand side of (a) agrees, then, with the bracket in (47). It is now a straightforward argument to show: *first,* that for n or i (or both) sufficiently large, the bracket is the dominant term in (a); *secondly,* that for i and n both bounded, a C exists. (47) then follows.

With a properly adjusted C we have the

COROLLARY. *For all n and all $i>0$,*

(48) $$|\lambda_{n+i}-\lambda_n| \geqq Ci(n+i)^{k-1} \quad (C>0, \textit{ independent of } n \textit{ and } i).$$

The coefficients in the series

$$\mathcal{D}_n(t) = \sum_{i=n}^{\infty} d_{ni}t^i$$

are given (see (13)) by*

(49) $$(\lambda_s-\lambda_n)d_s+\alpha_{s,s-1}d_{s-1}+\alpha_{s,s-2}d_{s-2}+\cdots+\alpha_{s,s-k}d_{s-k}=0, \qquad s=n+1,n+2,\cdots,$$

with $d_n=1$. Let S be a positive number such that

(b) $$|l_{ij}| \leqq S, \quad 0\leqq i, \quad j\leqq k.$$

Then

LEMMA 6. *For all n and $i(i>0)$ we have, with C as in* (48),

(50) $$|d_{n,n+i}| \leqq \frac{1}{i!}\left(\frac{Sk}{C}\right)\left(1+\frac{Sk}{C}\right)^{i-1}.$$

(50) is readily established (by use of (48)) for $i=1$, 2. Assuming it to be true up to $i-1$, we shall prove it true for i by induction, as follows:

* For simplicity we use d_s for d_{ns}.

$$|d_{n+i}| \leqq \frac{1}{|\lambda_{n+i} - \lambda_n|}[|\alpha_{n+i,n+i-1}d_{n+i-1}| + \cdots + |\alpha_{n+i,n+i-k}d_{n+i-k}|]$$

$$\leqq \frac{Sk}{Ci(n+i)^{k-1}}\left[(n+i-1)^{k-1}\frac{db^{i-2}}{(i-1)!} + (n+i-2)^{k-2}\frac{db^{i-3}}{(i-2)!} + \cdots + (n+i-k)^0\frac{db^{i-k-1}}{(i-k)!}\right],$$

where $a = (Sk/C)$, $b = 1 + (Sk/C)$;

$$|d_{n+i}| \leqq \frac{a^2b^{i-k-1}}{i!}\left[b^{k-1} + \frac{(i-1)}{n+i}b^{k-2} + \frac{(i-1)(i-2)}{(n+i)^2}b^{k-3} + \cdots + \frac{(i-1)\cdots(i-k+1)}{(n+i)^{k-1}}\right]$$

$$\leqq \frac{a^2b^{i-k-1}}{i!}[b^{k-1} + b^{k-2} + \cdots + 1] \leqq \frac{b^k}{b-1}\cdot\frac{a^2b^{i-k-1}}{i!} = \frac{ab^{i-1}}{i!},$$

and this is (50).

Consequently,

LEMMA 7.

$$(51)\quad \mathcal{D}_n(t) \ll t^n + \frac{a}{1!}t^{n+1} + \frac{ab}{2!}t^{n+2} + \cdots + \frac{ab^{i-1}}{i!}t^{n+i} + \cdots \ll pt^ne^{bt},$$

a, b, $p = \max(1, a)$ independent of n.

We now turn to $y_n(x)$. Its coefficients satisfy (as we see from (5)) the equations

$$(52)\quad \begin{aligned}(\lambda_s - \lambda_n)y_s + \sigma_{s,s+1}y_{s+1} + \sigma_{s,s+2}y_{s+2} + \cdots + \sigma_{s,s+k}y_{s+k} &= 0\\ (s = 0, 1, \cdots, n-1),&\end{aligned}$$

with $y_n = 1$.

LEMMA 8. *The coefficients of $y_n(x)$ satisfy the inequality*

$$(53)\qquad |y_{n,n-i}| \leqq \frac{n(n-1)\cdots(n-i+1)}{i!}\left(\frac{Sk}{C}\right)\left(1 + \frac{Sk}{C}\right)^{i-1}$$

for all n and i ($i \leqq n$).

The proof, by induction, differs very little from that of Lemma 6, and may be omitted. From (53) we get

LEMMA 9.

$$(54)\qquad \begin{aligned} y_n(x) &\ll x^n + \frac{n}{1!} a x^{n-1} + \frac{n(n-1)}{2!} ab x^{n-2} + \cdots + \frac{n!}{n!} ab^{n-1} \\ &\ll q\left[x^n + \frac{n}{1!} x^{n-1} b + \cdots + \frac{n!}{n!} b^n \right] \ll q(x+b)^n, \end{aligned}$$

$q = \max(1, a/b)$ *independent of* n, *so that for all* n *and* x,

$$(55)\qquad |y_n(x)| \leqq q(|x| + b)^n.$$

Combining Lemmas 7, 9:

$$|y_n(x)\mathcal{D}_n(t)| \leqq pqe^{b|t|}[|t|(|x| + b)]^n \text{ (for all } n, x, t).$$

Hence we have

THEOREM 8. *The series* $\sum_0^\infty y_n(x)\mathcal{D}_n(t)/n!$ *converges uniformly in every bounded* x, t *region, and represents an entire function in the two variables* x, t.

COROLLARY. *The series* $\sum_0^\infty \{y_n(x)\mathcal{D}_n(t)/[(\lambda_n - \lambda)n!]\}$ *converges uniformly in every bounded* x, t, λ *region (the points* $\lambda = \lambda_0, \lambda_1, \cdots$ *being deleted).*

The above two series are the right hand members of (45, 46). It remains to prove that they represent the corresponding left hand members. Let $H(t, x; \lambda)$ denote the sum of the series in the above corollary. H and Q have then the same principal parts at $\lambda = \lambda_0, \lambda_1, \cdots$ so that $Q - H$ is an entire function in all three variables. On applying the operator L_λ term-wise (as we may) to the series H, we get

$$(a)\qquad L_\lambda[Q(t, x; \lambda) - H(t, x; \lambda)] = e^{tx} - \sum_0^\infty y_n(x)\mathcal{D}_n(t)/n!.$$

Hence the left hand member is independent of λ, and represents a function $C(t, x)$ that is entire in t and x: $C(t, x) = \sum_{m,n=0}^\infty c_{mn} x^m t^n$. The right hand member of (a) has *zero* as coefficient of $x^m t^n$, $m \geqq n$, so that $c_{mn} = 0$, $m \geqq n$. Hence

$$(b)\qquad C(t, x) = \sum_{n=1}^\infty c_n(x) t^n,$$

where $c_n(x)$ is a polynomial of degree not exceeding $n-1$.

Now e^{tx} and $\sum_0^\infty y_n(x)\mathcal{D}_n(t)/n!$ are self-dual functions; the same is then true of $C(t, x)$:

$$(c)\qquad L[C(t, x)] = \mathcal{L}[C(t, x)].$$

On substituting into (c) the series (b), and equating coefficients of like powers of t, we obtain the equations

$$\text{(d)}\qquad \begin{aligned} &L_0(x)c_n(x) + L_1(x)c_n'(x) + \cdots + L_k(x)c_n^{(k)}(x) \\ &= \lambda_n c_n(x) + \alpha_{n,n-1}c_{n-1}(x) + \alpha_{n,n-2}c_{n-2}(x) + \cdots + \alpha_{n,n-k}c_{n-k}(x) \\ &\qquad\qquad (n = 1, 2, \cdots). \end{aligned}$$

It is at once verified that† $c_1(x)=0$. Assume that $c_1(x)=c_2(x)= \cdots =c_{n-1}(x)=0$. We shall show that $c_n(x)=0$. On our induction assumption, (d) reduces to

$$\text{(e)}\qquad L[c_n(x)] = \lambda_n c_n(x).$$

$c_n(x)$ is an entire function, so that by Theorem 1, $c_n(x)\equiv a_n y_n(x)$, a_n a constant. But $c_n(x)$ is of degree less than n; hence $a_n=0$, and $c_n(x)=0$. That is, $C(t, x)\equiv 0$, and the right hand member of (a) is zero.

We have just established (46). Equation (a) then gives us

$$\text{(a}'\text{)}\qquad L_\lambda[Q(t, x; \lambda) - H(t, x; \lambda)] = 0,$$

where $Q-H$ is entire in t, x, λ. By Theorem 1, (a′) has an entire function solution ($\not\equiv 0$) if and only if $\lambda=\lambda_0, \lambda_1, \cdots$. Hence $Q-H\equiv 0$, $\lambda\neq\lambda_0, \lambda_1, \cdots$, and by continuity $Q-H\equiv 0$ for all t, x, λ. That is, $Q\equiv H$, and this is (45). We thus have

THEOREM 9. *The two series*

$$\text{(45)}\quad Q(t, x; \lambda) = \sum_{n=0}^{\infty} \frac{y_n(x)\mathcal{D}_n(t)}{(\lambda_n - \lambda)n!}, \qquad \text{(46)}\quad e^{tx} = \sum_{n=0}^{\infty} y_n(x)\mathcal{D}_n(t)/n!$$

are valid for all t, x, λ ($\lambda_0, \lambda_1, \cdots$ deleted).

COROLLARY. *The expansion*

$$\text{(56)}\qquad P(t, x; \lambda) = \sum_{n=0}^{\infty}(\lambda_n - \lambda)y_n(x)\mathcal{D}_n(t)/n!$$

is uniformly convergent in every bounded t, x, λ region, thus representing an entire function in all three variables. (See (26).)

5. **Further inequalities; expansions in $\mathcal{D}_n(t)$, $Y_n(x)$.** In the present section we investigate the question of expansions of functions in terms of the two sets $\{\mathcal{D}_n(t)\}$, $\{Y_n(x)\}$ (introduced in (19)). For this we require some further inequalities. In the equation

† For $c_1(x)\equiv$ constant $=c$, say. Then $l_{00}c=(l_{00}+l_{11})c$, so that $cl_{11}=0$. Now if $l_{11}=0$, then $\lambda_1=\lambda_0$, which contradicts our assumption ($\lambda_m\neq\lambda_n$, $m\neq n$). Hence $l_{11}\neq 0$, and $c=0$.

(4′) $$L[y_n(x)] = \lambda_n y_n(x)$$

make the transformation $x = x^* - \gamma$. On setting $y_n^*(x^*) = y_n(x)$ we have

(4″) $$L^*[y_n^*(x^*)] = \lambda_n y_n^*(x^*),$$

where L^* is an operator similar to L, $L_i^*(x^*)$ being equal to $L_i(x)$. In particular, $l_{kk}^* = l_{kk}$, $l_{k,k-1}^* = -kl_{kk}\gamma + l_{k,k-1}$.

If we denote by $\{\mathcal{D}_n^*(t)\}$ the set of functions corresponding to $\{y_n^*(x^*)\}$, then by (46),

(46′) $$e^{tx^*} = \sum_{n=0}^{\infty} y_n^*(x^*)\mathcal{D}_n^*(t)/n!.$$

But $y_n^*(x^*) = y_n(x)$, $e^{tx^*} = e^{tx}\cdot e^{\gamma t}$. Hence $\mathcal{D}_n(t)$ is transformed into

(57) $$\mathcal{D}_n^*(t) \equiv e^{\gamma t}\mathcal{D}_n(t).$$

Choose

(58) $$\gamma = l_{k,k-1}/kl_{kk}.$$

Then $l_{k,k-1}^* = 0$; i.e., the sum of all the zeros of $L_k^*(x^*)$ is zero. There is clearly no loss in generality if we go from (4) to (4″) for the choice of γ in (58). Then, dropping asterisks, *we shall henceforth suppose that in* (4′), $l_{k,k-1} = 0$.

LEMMA 10. *For all n and $s(s>0)$ we have*

(59) $$|d_{n,n+s}| \leqq \frac{1}{n}\,\frac{h^s}{s!}, \quad d_{nn} = 1,$$

h being independent of n and s.

To show this, substitute in equations (49) for the d's the numbers r_i defined by

(a) $$r_{n+s} = nd_{n+s},\ s = 0, 1, \cdots, r_n = n.$$

This gives

(b) $$(\lambda_{n+s} - \lambda_n)r_{n+s} + \alpha_{n+s,n+s-1}r_{n+s-1} + \cdots + \alpha_{n+s,n+s-k}r_{n+s-k} = 0 \qquad (s = 1, 2, \cdots).$$

Solving for r_{n+s}, and using the values of the α_{ij}'s (given by (14)) as well as the relation $l_{k,k-1} = 0$ and the inequalities (48), we find that

(c) $$r_n = n,\ |r_{n+s}| \leqq (h/s)\left[\frac{|r_{n+s-1}|}{n+s} + \frac{|r_{n+s-2}|}{n+s} + \frac{|r_{n+s-3}|}{(n+s)^2} + \cdots + \frac{|r_{n+s-k}|}{(n+s)^{k-1}}\right],$$

where $h \geqq Sk/C$. Choose $h = \max(2, Sk/C)$. On setting

$$\text{(d)}\quad t_n = n,\ t_{n+s} = (h/s)\left[\frac{t_{n+s-1}}{n+s} + \frac{t_{n+s-2}}{(n+s)} + \frac{t_{n+s-3}}{(n+s)^2} + \cdots + \frac{t_{n+s-k}}{(n+s)^{k-1}}\right],$$

we have $t_{n+s} \geqq |r_{n+s}|$.

A simple calculation gives us $t_{n+1} \leqq h/1!$, $t_{n+2} \leqq h^2/2!$; we shall establish the relation

$$\text{(e)}\qquad t_{n+s} \leqq h^s/s!,\ s > 0,$$

by induction, assuming it true up to $s-1$. For

$$\begin{aligned} t_{n+s} &\leqq \frac{h}{s}\left[\frac{h^{s-1}}{(s-1)!(n+s)} + \frac{h^{s-2}}{(s-2)!(n+s)} + \cdots + \frac{h^{s-k}}{(s-k)!(n+s)^{k-1}}\right] \\ &\leqq \frac{h^s}{s!}\,\frac{1}{n+s}\left[1 + \frac{s-1}{h}\left\{1 + \frac{s-2}{h(n+s)} + \frac{(s-2)(s-3)}{h^2(n+s)^2}\right.\right. \\ &\qquad \left.\left. + \cdots + \frac{(s-2)\cdots(s-k+1)}{h^{k-2}(n+s)^{k-2}}\right\}\right] \\ &\leqq \frac{h^s}{s!}\,\frac{1}{n+s}\left[1 + \frac{2(s-1)}{h}\right] \leqq \frac{h^s}{s!}, \end{aligned}$$

which is (e). Then, $|r_{n+s}| \leqq h^s/s!$, and from this (59) follows. From (59) we get

THEOREM 10. *$\mathcal{D}_n(t)$ is asymptotically given by*

$$\text{(60)}\qquad \mathcal{D}_n(t) = t^n[1 + A_n(t)]$$

where

$$\text{(61)}\qquad |A_n(t)| \leqq e^{h|t|}/n.$$

THEOREM 11. *Let $C(t) = \sum_0^\infty c_n t^n$ have r as its radius of convergence. Then the series*† *$C^*(t) = \sum_0^\infty c_n \mathcal{D}_n(t)$*

(a) *converges absolutely at every point in $|t| < r$;*

(b) *converges uniformly in $|t| \leqq r' < r$, r' arbitrary;*

(c) *diverges in $|t| > r$.*

In particular, $\mathcal{D}_n(t)$-expansions have circles, center at origin, as their regions of convergence.

In fact, for n sufficiently large,

$$\tfrac{1}{2}|c_n t^n| \leqq |c_n \mathcal{D}_n(t)| = |c_n t^n| \cdot |1 + A_n(t)| \leqq 2|c_n t^n|.$$

† If $r = 0$, the only point of convergence for the $C^*(t)$-series is $t = 0$.

In $C^*(t)=\sum_0^\infty c_n\mathcal{D}_n(t)$, let $\limsup |c_n|^{1/n}=\sigma<\infty$, so that the series has a radius of convergence $r=1/\sigma>0$. We may expand $C^*(t)$ in a power series in t: $C^*(t)=\sum_0^\infty c_n^* t^n$, with

$$\text{(i)}\qquad c_n^* = c_0d_{0n} + c_1d_{1n} + \cdots + c_nd_{nn} \qquad (n = 0, 1, \cdots).$$

Since, by hypothesis, $|c_n| \leqq A(\sigma+\epsilon)^n$, $\epsilon>0$, $A=A_\epsilon$, we have (see (50))

$$|c_n^*| \leqq A[a(1+a)^{n-1}/n! + (\sigma+\epsilon)a(1+a)^{n-2}/(n-1)! + \cdots + (\sigma+\epsilon)^{n-1}a/1! + (\sigma+\epsilon)^n],$$

$a=Sk/C$. That is,

$$|c_n^*| \leqq A(\sigma+\epsilon)^n\left[1+\frac{a}{\sigma+\epsilon}\left\{\frac{1}{1!}+\frac{1}{2!}\left(\frac{1+a}{\sigma+\epsilon}\right)+\cdots+\frac{1}{n!}\left(\frac{1+a}{\sigma+\epsilon}\right)^{n-1}\right\}\right]$$

$$\leqq A\left[1+\frac{a}{\sigma+\epsilon}e^{(1+a)/(\sigma+\epsilon)}\right](\sigma+\epsilon)^n \leqq B(\sigma+\epsilon)^n,\ B \text{ independent of } n.$$

Therefore $\limsup |c_n^*|^{1/n}\leqq\sigma$, and we have

LEMMA 11. *If $C^*(t)=\sum_0^\infty c_n\mathcal{D}_n(t)$ has the radius of convergence r, then $C^*(t)=\sum_0^\infty c_n^*t^n$ has a radius of convergence at least as great as r.*

The converse theorem is also true. Its proof, which is not so immediate, can be made to depend on inequalities regarding the functions $Y_n(x)$ of (19):

$$\text{(19)}\qquad Y_n(x) = 0!y_{n0} + 1!y_{n1}x + \cdots + n!y_{nn}x^n.$$

Set $z_{ni}=i!y_{ni}/n!$. Then, by (5),

$$\text{(62)}\qquad (\lambda_s-\lambda_n)(s+k)\cdots(s+1)z_s + \sigma_{s,s+1}(s+k)\cdots(s+2)z_{s+1} + \cdots + \sigma_{s,s+k}z_{s+k} = 0 \quad (s=0,1,\cdots),$$

with $z_n=1$. We find, using (48), that

$$|z_{n-1}| \leqq \frac{h}{1!n},\quad |z_{n-2}| \leqq \frac{h^2}{2!n},\quad n>1,\ h=\max(2, Sk/C),$$

and an induction gives us (compare Lemma 10)

LEMMA 12. *For all $n>1$ and all i $(0<i\leqq n)$,*

$$\text{(ii)}\qquad |z_{n,n-i}| \leqq h^i/(i!n).$$

We can write

$$Y_n(x)/n! = x^n[z_n + z_{n-1}/x + \cdots + z_0/x^n]$$

$$= x^n\left[1+\frac{1}{n}\left\{\frac{1}{1!}\left(\frac{h}{x}\right)\delta_{n1}+\frac{1}{2!}\left(\frac{h}{x}\right)^2\delta_{n2}+\cdots+\frac{1}{n!}\left(\frac{h}{x}\right)^n\delta_{nn}\right\}\right],$$

where $|\delta_{ni}| \leqq 1$ for all n and i. This gives us

THEOREM 12. *$Y_n(x)$ has the asymptotic form*

$$Y_n(x)/n! = x^n[1 + B_n(x)/n] \tag{63}$$

where†

$$|B_n(x)| \leqq e^{h/|x|}. \tag{64}$$

Let us now consider the expansion (46) for e^{tx}. This is (in the variable x) the Borel entire function associated with $1/(1-tx)$, and $y_n(x)$ is the Borel function associated with $Y_n(x)$, thus suggesting the expansions

$$1/(1 - tx) = \sum_{n=0}^{\infty} Y_n(x)\mathcal{D}_n(t)/n! \equiv T(t, x), \tag{a}$$

$$\frac{1}{x - t} = \sum_{n=0}^{\infty} \frac{Y_n(1/x)\mathcal{D}_n(t)}{x(n!)}. \tag{65}$$

To verify this and determine the region of validity of (a, 65), we appeal to relations (60) and (63), which show us that $T(t, x)$ is analytic in any region for which $|tx| < 1$, and that the series for $T(t, x)$ converges uniformly for $|tx| \leqq \alpha < 1$. Let x trace the circle $|x| = \delta > 0$. The series in question then converges uniformly for $|t| \leqq \alpha/\delta$, and we may multiply it by $e^{x/u}/u$ and integrate term-wise with respect to u around Γ: $|u| = \delta$:

$$\frac{1}{2\pi i}\int_\Gamma \frac{T(t, u)}{u} e^{x/u} du = \sum_0^\infty \frac{\mathcal{D}_n(t)}{n!}\left\{\frac{1}{2\pi i}\int_\Gamma \frac{Y_n(u)}{u} e^{x/u} du\right\}, \quad = e^{tx}$$

since

$$\frac{1}{2\pi i}\int_\Gamma \frac{Y_n(u)}{u} e^{x/u} du = y_n(x).$$

If we now expand $T(t, u)$ in a power series about $u = 0$ (as we may):

$$T(t, u) = \sum_0^\infty T_n(t) u^n,$$

we find that $T_n(t) = t^n$, so that (a) is true for $|tx| < 1$. Therefore (65) holds. Now by (63), lim sup $|Y_n(1/x)/n!|^{1/n} = 1/|x|$; whence from Theorem 11 follows

† For $x = 0$, $|Y_n(x)/n!| = |z_{n0}| \leqq h^n/(n!\, n)$.

THEOREM 13. *For every x the series* (65) *has the interior of the circle* $|t| = |x|$ *as its region*† *of convergence. In every region* $|t| \leqq \rho < |x|$ *the convergence is uniform.*

If $f(t) = \sum_0^\infty f_n t^n$ is analytic in $|t| < r$, we get from (65), by Cauchy's integral formula:

$$f(t) = \sum_{n=0}^{\infty} \left\{ \frac{1}{2\pi i} \int_{|x|=r'} \frac{Y_n(1/x)f(x)}{x(n!)} dx \right\} \mathcal{D}_n(t), \tag{66}$$

valid for $|t| < r' < r$. But r' can be chosen as close to r as we desire; hence (66) is true for all $|t| < r$, and is uniformly convergent in every closed region in $|t| < r$. This is the converse of Lemma 11. Combining the two we have

THEOREM 14. *The function* $f(t)$ *has a convergent* $\mathcal{D}_n(t)$*-expansion if and only if it is analytic about* $t = 0$; *and its* $\mathcal{D}_n(t)$*-expansion and its power series expansion have the same radius of convergence.*

A $\mathcal{D}_n(t)$*-expansion is unique.* For if $\sum_0^\infty c_n \mathcal{D}_n(t)$ converges, it converges uniformly (Theorem 11), and we may write $\sum_0^\infty c_n \mathcal{D}_n(t) = \sum_0^\infty c_n^* t^n$, where the c_n^* are given by (i). If the c_n^*'s are given, these equations (i) determine $c_0, c_1, \cdots$ uniquely.

Let us sum up our theorems on $\mathcal{D}_n(t)$-series:

THEOREM 15. *The series* $\sum_0^\infty c_n \mathcal{D}_n(t)$ *has the single point of convergence* $t = 0$ *if and only if* $\limsup |c_n|^{1/n} = \infty$. *If* $\limsup |c_n|^{1/n} = \sigma < \infty$, *then the series converges throughout* $|t| < 1/\sigma$ *and diverges throughout* $|t| > 1/\sigma$. *In* $|t| < 1/\sigma$ *the convergence is absolute, and in every closed region in* $|t| < 1/\sigma$ *it is uniform. If* $f(t)$ *denotes the sum of the series, then* $f(t)$ *is analytic in* $|t| < 1/\sigma$, *and* $1/\sigma$ *is the radius of convergence of* $f(t) = \sum_0^\infty f_n t^n$, *i.e., the* $\mathcal{D}_n(t)$*-series and the power series for the same function have the same radius of convergence. Furthermore, a* $\mathcal{D}_n(t)$*-expansion is unique, and the coefficients are given by*

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(x)}{x} \cdot \frac{Y_n(1/x)}{n!} dx, \tag{67}$$

C being a contour about $x = 0$ *and lying in* $|x| < 1/\sigma$.

We turn now to $Y_n(x)$-expansions. From (60, 63, 46) we readily deduce

LEMMA 13. *The function* $1/(t-x)$ *has the expansion*

$$\frac{1}{t - x} = \sum_{n=0}^{\infty} \frac{Y_n(x)\mathcal{D}_n(1/t)}{t(n!)}, \tag{68}$$

† Points of the boundary may be points of convergence.

which for a given t has the interior of the circle $|x| = |t|$ *as its region of convergence. In every region* $|x| \leqq \rho < |t|$ *the convergence is uniform.*

In ways analogous to those used for Theorem 15, we can establish

THEOREM 16. *Everything said*† *of* $\mathcal{D}_n(t)$*-expansions in Theorem* 15 *holds for* $\{Y_n(x)/n!\}$*-expansions, with the modification that the* c_n*'s in* $\sum_0^\infty c_n Y_n(x)/n!$ *are now given by*

$$(69) \qquad c_n = \frac{1}{2\pi i}\int_C \frac{f(t)}{t}\,\mathcal{D}_n(1/t)dt,$$

C being a contour about $t=0$ *and lying in* $|t| < 1/\sigma$.

6. **Biorthogonality relations; differential equations for** $\Delta_n(t)$, $Y_n(x)$. In the equation

$$(18) \qquad \mathcal{D}_n(t) = \frac{1}{2\pi i}\int_C \frac{e^{ut}}{u}\Delta_n(1/u)du,$$

C being a contour surrounding $u=0$ and lying outside of $|u| = \rho$, replace e^{ut} by its expansion $\sum_{s=0}^\infty y_s(u)\mathcal{D}_s(t)/s!$, which converges uniformly on C. This gives us

$$\mathcal{D}_n(t) = \sum_{s=0}^\infty \left\{\frac{1}{2\pi i}\int_C \frac{y_s(u)\Delta_n(1/u)}{s!u}du\right\}\mathcal{D}_s(t),$$

whence by uniqueness of $\mathcal{D}_n(t)$-expansions we have

THEOREM 17. *The functions* $\{y_n\}$, $\{\Delta_n\}$ *are biorthogonal in the following sense*:

$$(70) \qquad \frac{1}{2\pi i}\int_C \frac{y_s(u)\Delta_n(1/u)}{n!u}du = \begin{cases} 0, & s \neq n, \\ 1, & s = n. \end{cases}$$

If we start with the relation

$$(20) \qquad y_n(x) = \frac{1}{2\pi i}\int_\Gamma \frac{e^{xu}}{u}Y_n(1/u)du,$$

Γ being a contour around $u=0$, we obtain the uniformly convergent expansion

$$(a) \qquad y_n(x) = \sum_{s=0}^\infty \left\{\frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{D}_s(u)Y_n(1/u)}{s!u}du\right\}y_s(x).$$

As we have not established uniqueness of $y_n(x)$-expansions, we cannot at once conclude that the brace in (a) is zero or one. But this can be proved in the following way.

† We must of course except the conclusion of Theorem 15 for the case $\limsup_{n\to\infty} |c_n|^{1/n} = \infty$.

Denote by c_{ns} the brace in (a), so that $y_n(u)=\sum_{s=0}^{\infty}c_{ns}y_s(u)$, uniformly convergent for all bounded u. Multiply both members by $\Delta_r(1/u)/(n!u)$ and integrate term-wise over the contour C of (70). This gives us, by (70),

$$\delta_{nr} = \sum_{s=0}^{\infty} c_{ns}s!\delta_{sr}/n!; \quad \text{i.e.}, \delta_{nr} = \frac{c_{nr}r!}{n!},$$

or $c_{nr}=0$, $r\neq n$; $c_{nn}=1$. Hence we have

THEOREM 18. *The functions* $\{\mathcal{D}_n\}$, $\{Y_n\}$ *are biorthogonal in the sense*

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{\mathcal{D}_s(u)Y_n(1/u)}{n!u}du = \begin{cases}0, & s\neq n,\\ 1, & s=n.\end{cases} \tag{71}$$

By means of (70) we can show that a $y_n(x)$-*expansion is unique if it converges uniformly in a region that contains the region* $|x|\leqq\rho+\epsilon,\epsilon>0$ sufficiently small. Here ρ is, as before, the maximum absolute value of the zeros of $L_k(x)$. This is equivalent to saying that if the function *zero* has a y_n-expansion that is uniformly convergent in $|x|\leqq\rho+\epsilon$, then the coefficients in the expansion are all zero. This result follows on multiplying the series in question through by $(1/u)(1/n!)\Delta_n(1/u)$ and integrating over C, using (70).

We can sharpen this conclusion by further considering the functions $\Delta_n(t)$. We know $\mathcal{D}_n(t)$ is the Borel entire function corresponding to $\Delta_n(t)$. In general,

DEFINITION. *If the two functions* $A(t)=\sum_0^{\infty}a_nt^n$, $B(t)=\sum_0^{\infty}n!a_nt^n$ *are analytic about* $t=0$, *then* $A(t)$ *is the Borel entire function associated with* $B(t)$, *and we write* $A(t)=BEF\{B(t)\}$.

It is easily established that

LEMMA 14. *If* $A(t)=BEF\{B(t)\}$, *then* $A'(t)=BEF\{(B(t)-B(0))/t\}$.

LEMMA 15. *For all* $0\leqq j\leqq i$,

$$t^i\mathcal{D}_n^{(i)}(t) = BEF\left\{t^i\frac{d^i}{dt^i}(t^{i-j}\Delta_n(t))\right\}. \tag{72}$$

This can be established by a straightforward induction argument, using Lemma 14; the proof may therefore be omitted.

Since $\mathcal{D}_n(t)$ satisfies the differential equation

$$\mathcal{L}[\mathcal{D}_n(t)] \equiv \mathcal{L}_0(t)\mathcal{D}_n(t) + \mathcal{L}_1(t)\mathcal{D}_n'(t) + \cdots + \mathcal{L}_k(t)\mathcal{D}_n^{(k)}(t) = \lambda_n\mathcal{D}_n(t),$$

where $\mathcal{L}_i(t)$ is a polynomial having no term t^k with $k<i$, we may apply Lemma 15. It gives us

THEOREM 19. *The functions $\Delta_n(t)$ satisfy the kth-order linear homogeneous differential equation*

$$\sum_{i=j}^{k}\sum_{j=0}^{k}\left\{l_{ij}t^i\frac{d^i}{dt^i}(t^{i-j}\Delta_n(t))\right\} = \lambda_n\Delta_n(t). \tag{73}$$

Here the coefficients of the various derivatives of $\Delta_n(t)$ are polynomials in t, that of $\Delta_n^{(k)}(t)$ being

$$t^k(l_{k0}t^k + l_{k1}t^{k-1} + \cdots + l_{kk}) = t^{2k}L_k(1/t).$$

COROLLARY 1. *The only possible singularities (in the finite plane) of the functions $\Delta_n(t)$ are at the reciprocals of the zeros of $L_k(t)$.*

$\Delta_n(1/t)$ has then only the zeros of $L_k(t)$, and the origin, as possible singularities. From this follows

COROLLARY 2. *In relations (18) and (70), the contour C may be chosen as any contour which has the origin and all the zeros of $L_k(u)$ in its interior.*

By the argument used just before Lemma 14, applied to equation (70) with a contour C of Corollary 2, we have

THEOREM 20. *A $y_n(x)$-expansion is unique† if it converges uniformly in a simply-connected open region $\mathfrak{R}$ that contains the origin and all the zeros of $L_k(x)$.*

The numbers $\{\lambda_n\}$ are the characteristic numbers for our original equations (1), (12). It is natural to inquire if they have like significance for equation (73), regarded independently of its origin. The answer, in the affirmative, is given by

THEOREM 21. *The differential equation*

$$\sum_{i=j}^{k}\sum_{j=0}^{k}\left\{l_{ij}t^i\frac{d^i}{dt^i}(t^{i-j}\Delta(t))\right\} = \lambda\Delta(t) \tag{73'}$$

has a formal power series solution about the origin if and only if $\lambda=\lambda_0, \lambda_1, \cdots$, and when $\lambda=\lambda_n$, there is a unique solution (to within an arbitrary constant multiplier); this solution converges about $t=0$, and is, in fact, the function $\Delta_n(t)$.

To prove this, assume the expansion $\Delta(t)=\sum_0^\infty a_nt^n$. On substituting into (73′) and equating coefficients, the values $\lambda_0, \lambda_1, \cdots$ are found to be the only possible ones, and these yield unique solutions. The remainder of the theorem follows from the fact that $\Delta_n(t)$ *is* a solution for $\lambda=\lambda_n$.

† That is, if two such expansions (uniformly convergent in $\mathfrak{R}$) represent the same function, corresponding coefficients are equal.

We now turn to the functions $\{Y_n(x)\}$. From equation (4) for $y_n(x)$ and the property that $y_n(x)=BEF\{Y_n(x)\}$, we derive the corresponding differential equation for $Y_n(x)$. Unlike the case for $\Delta_n(t)$, however, the new equation is of infinite order. In fact, we have

$$\text{(a)} \qquad Y_n(x) = Y_0 + Y_1 x + \cdots + Y_n x^n,$$

where $Y_i = i! y_i$. (We write Y_i, y_i for Y_{ni}, y_{ni}.) From (a) follows

LEMMA 16. *The coefficients of $Y_n(x)$ are given by*†

$$\text{(b)} \quad Y_{n-i} = [1/(n-i)!][Y_n^{(n-i)}(x) - (x/1!)Y_n^{(n-i+1)}(x) + (x^2/2!)Y_n^{(n-i+2)}(x) - \cdots + (-1)^i(x^i/i!)Y_n^{(n)}(x)] \quad (i = 0, 1, \cdots, n).$$

To show this let $T_n(x)$ denote the right hand member of (b). Since $Y_n(x)$ is of degree n, $T_n(x)$ is unaltered if we add to it terms containing higher derivatives of $Y_n(x)$ than the nth. That is, we can write

$$T_n(x) = \left[\frac{1}{(n-i)!}\right] \sum_{s=0}^{\infty}(-1)^s\left(\frac{x^s}{s!}\right)Y_n^{(n-i+s)}(x),$$

the series being uniformly convergent in every bounded region. Letting C be a contour surrounding the origin, we have

$$Y_n(x) = \frac{1}{2\pi i}\int_C \frac{Y_n(u)}{u-x}\,du, \quad Y_n^{(n-i+s)}(x) = \frac{(n-i+s)!}{2\pi i}\int_C \frac{Y_n(u)}{(u-x)^{n-i+s+1}}\,du;$$

and on substituting into $T_n(x)$, we obtain

$$T_n(x) = \frac{1}{2\pi i}\int_C \frac{Y_n(u)}{(u-x)^{n-i+1}} \cdot\left\{\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}(n-i+1)(n-i+2)\cdots(n-i+s)\left(\frac{x}{u-x}\right)^s\right\}du.$$

Now the brace is the expansion of

$$\left[1+\left(\frac{x}{u-x}\right)\right]^{-(n-i+1)} = \left(\frac{u-x}{u}\right)^{n-i+1},$$

valid for $|x/(u-x)|<1$. We can choose C to satisfy this condition and also to surround the origin. Then,

$$T_n(x) = (1/(2\pi i))\int_C \{Y_n(u)/u^{n-i+1}\}\,du = Y_{n-i}.$$

That is, (b) holds.

† (b) is of course true for *any* polynomial of degree n.

Now let $0 \leqq j \leqq i$. From $y_n(x) = \sum_{i=0}^n y_i x_i$ we obtain

$$x^j y_n^{(i)}(x) = n(n-1)\cdots(n-i+1)y_n x^{n-i+j} + (n-1)\cdots(n-i)y_{n-1}x^{n-i+j-1} + \cdots + i!y_i x^j$$
$$= BEF\left\{\frac{(n-i+j)!}{(n-i)!}Y_n x^{n-i+j} + \frac{(n-i+j-1)!}{(n-i-1)!}Y_{n-1}x^{n-i+j-1} + \cdots + \frac{j!}{0!}Y_i x^j\right\};$$

and on using (b) of Lemma 16, this gives

$$\text{(c)}\quad x^j y_n^{(i)}(x) = BEF\{\theta_{n;ij}x^{n-i+j}Y_n^{(n)}(x) + \theta_{n-1;ij}x^{n-i+j-1}Y_n^{(n-1)}(x) + \cdots + \theta_{i;ij}x^j Y_n^{(i)}(x)\},$$

$$\text{(d)}\quad \theta_{p;ij} = \frac{(p-i+j)!}{(p-i)!p!0!} - \frac{(p-i+j-1)!}{(p-i-1)!(p-1)!1!} + \cdots + (-1)^{p-i}\frac{j!}{0!i!(p-i)!} \qquad (p = i, i+1, \cdots, n).$$

We readily get

LEMMA 17. *The quantity $\theta_{p;ij}$ is the coefficient of u^{p-i} in the power series expansion of $e^{-u}H_{ij}(u)$, where $H_{ij}(u)$ is the entire function*

$$H_{ij}(u) = \sum_{s=0}^{\infty} \frac{(j+s)!}{s!(i+s)!}u^s.$$

If we substitute the value of $x^j y_n^{(i)}(x)$ as given by (c) into equation (4), for $\lambda = \lambda_n$, we obtain

$$\text{(74)}\quad \sum_{i=0}^{k}\sum_{j=0}^{i} l_{ij}\{\theta_{n;ij}x^{n-i+j}Y_n^{(n)}(x) + \theta_{n-1;ij}x^{n-i+j-1}Y_n^{(n-1)}(x) + \cdots + \theta_{i;ij}x^j Y_n^{(i)}(x)\} = \lambda_n Y_n(x),$$

which is a linear homogeneous differential equation of order n for $Y_n(x)$. (74) can be written in the form

$$\text{(74')}\quad M_0(x)Y_n(x) + M_1(x)Y_n'(x) + \cdots + M_n(x)Y_n^{(n)}(x) = \lambda_n Y_n(x),$$

where

$$\text{(75)}\quad M_i(x) = \sum_{r=0}^{s}\{l_{r0}\theta_{i;r0}x^{i-r} + l_{r1}\theta_{i;r1}x^{i-r+1} + \cdots + l_{rr}\theta_{i;rr}x^i\},$$

with $s = k$ if $i \geqq k$, and $s = i$ if $i < k$.

Clearly, $M_i(x)$ is independent of n. Since $Y_n^{(s)}(x) \equiv 0$, $s > n$, we see that *the functions* $\{Y_n(x)\}$ *are solutions of the linear homogeneous differential equation of infinite order*

$$\sum_{s=0}^{\infty} M_s(x) Y^{(s)}(x) = \lambda Y(x), \tag{76}$$

where for $Y(x) = Y_n(x)$ *we have* $\lambda = \lambda_n$.

It is seen that $M_i(x)$ is a polynomial of degree not exceeding i, so that equation (76) belongs to the type considered in *Sets* (pp. 29–31).

THEOREM 22. *The only polynomial solutions of* (76) *are the polynomials* $\{Y_n(x)\}$, *and the only characteristic numbers are therefore* $\{\lambda = \lambda_n\}$.

For let $Y(x)$ be a polynomial satisfying (76) with the value $\lambda = \lambda'$. Then, since the relation between equations (4) and (76) can be traced in *both* directions, the polynomial $y(x)$ given by $y(x) = BEF\{Y(x)\}$ will satisfy (4) for $\lambda = \lambda'$. This can be true only if λ' is one of the numbers λ_n, and in this case we must have $y(x) \equiv y_n(x)$. Hence $Y(x) \equiv Y_n(x)$.

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